Subject Name: **Strength of materials**

Subject Code: **CE-3003**

Semester: **3rd**

UNIT-V

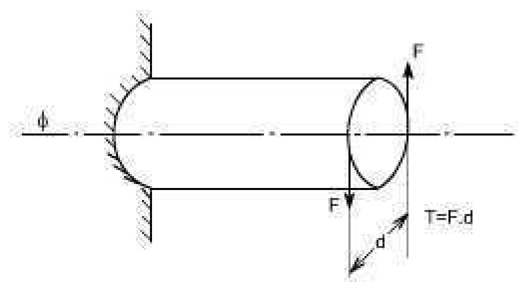
Torsion of Shafts: Concept of pure torsion, Torsion equation, Determination of shear stress and angle of twist of shafts of circular section, Torsion of solid and hollow circular shafts, Analyses of problems based on combined Bending and Torsion.

Unsymmetrical Bending: Principal moment of Inertia, Product of Inertia, Bending of a beam in a plane which is not a plane of, symmetry. Shear center; Curved beams: Pure bending of curved beams of rectangular, circular and trapezoidal sections, Stress distribution and position of neutral axis

Members Subjected to Torsional Loads

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross – section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

GENERATION OF SHEAR STRESSES

The physical understanding of the phenomena of setting up of shear stresses in a shaft subjected to a torsion may be understood from the figure 1-3.

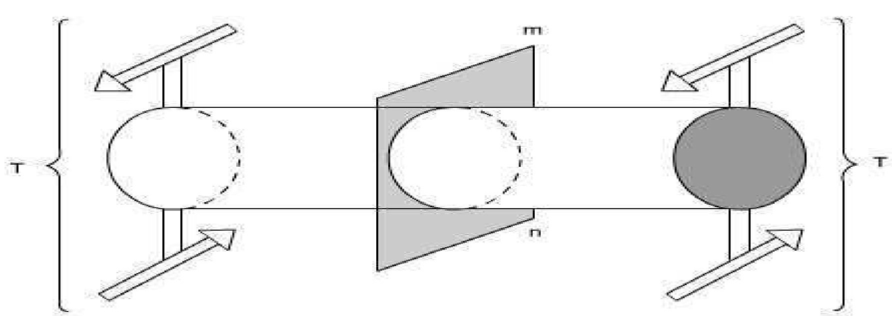


Fig 1: Here the cylindrical member or a shaft is in static equilibrium where T is the resultant external torque acting on the member. Let the member be imagined to be cut by some imaginary plane 'mn'.

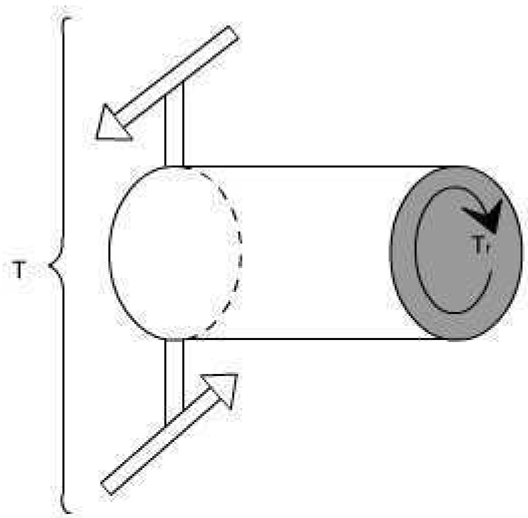


Fig 2: When the plane 'mn' cuts remove the portion on R.H.S. and we get a fig 2. Now since the entire member is in equilibrium, therefore, each portion must be in equilibrium. Thus, the member is in equilibrium under the action of resultant external torque T and developed resisting Torque T_r .

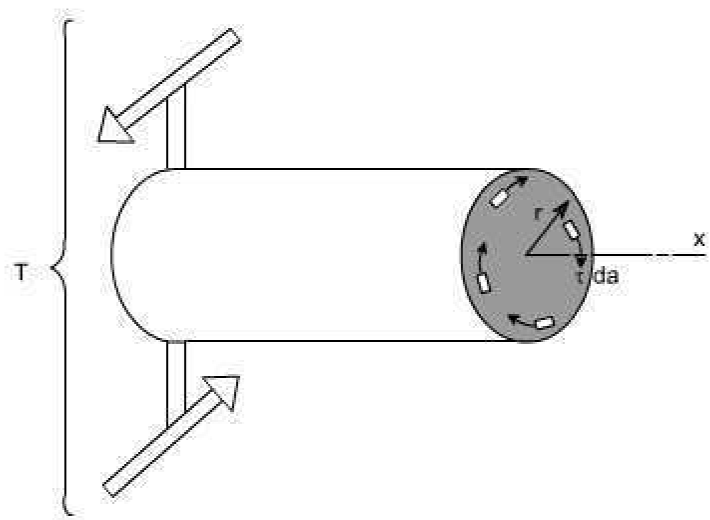


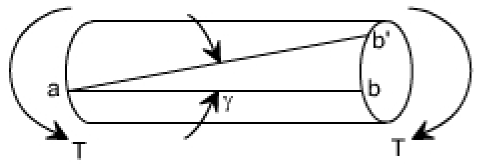
Fig 3: The Figure shows that how the resisting torque T_r is developed. The resisting torque T_r is produced by virtue of an infinites mal shear forces acting on the plane perpendicular to the axis of the shaft. Obviously such shear forces would be developed by virtue of sheer stresses.

Therefore we can say that when a particular member (say shaft in this case) is subjected to a torque, the result would be that on any element there will be shear stresses acting. While on other faces the complementary sheer forces come into picture. Thus, we can say that when a member is subjected to torque, an element of this member will be subjected to a state of pure shear.

Shaft: The shafts are the machine elements which are used to transmit power in machines.

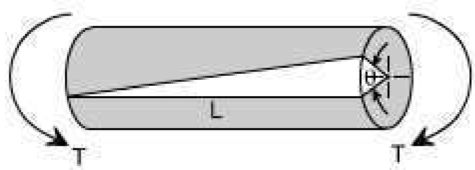
Twisting Moment: The twisting moment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a – b is marked on the surface of the unloaded bar, then after the twisting moment 'T' has been applied this line moves to ab'. The angle 'γ' measured in radians, between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.



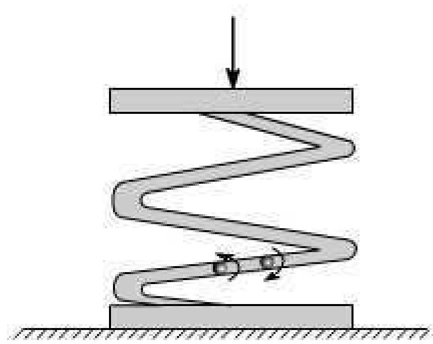
Modulus of Elasticity in shear: The ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol $G = \frac{\tau}{\gamma}$

Angle of Twist: If a shaft of length L is subjected to a constant twisting moment T along its length, than the angle θ through which one end of the bar will twist relative to the other is known is the angle of twist.



- Despite the differences in the forms of loading, we see that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.
  In torsion the members are subjected to moments (couples) in planes normal to their axes.
- For the purpose of designing a circular shaft to withstand a given torque, we must develop an equation giving the relation between twisting moment, maximum shear stress produced, and a quantity representing the size and shape of the cross-sectional area of the shaft.

Not all torsion problems, involve rotating machinery, however, for example some types of vehicle suspension system employ torsional springs. Indeed, even coil springs are really curved members in torsion as shown in figure.



- Many torque carrying engineering members are cylindrical in shape. Examples are drive shafts, bolts and screw drivers.

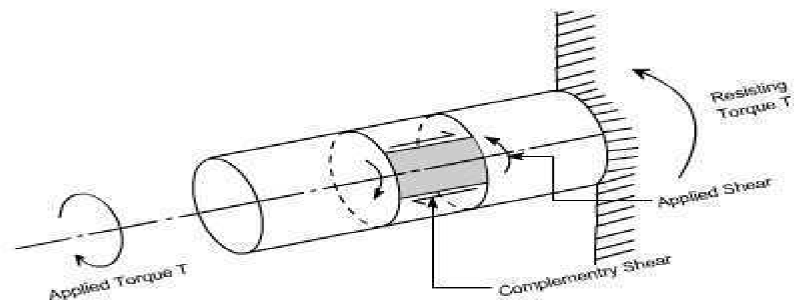
Simple Torsion Theory or Development of Torsion Formula : Here we are basically interested to derive an equation between the relevant parameters

Relationship in Torsion: $\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$

1st Term: It refers to applied loading ad a property of section, which in the instance is the polar second moment of area.

2nd Term: This refers to stress, and the stress increases as the distance from the axis increases.

3rd Term: it refers to the deformation and contains the terms modulus of rigidity & combined which is equivalent to strain for the purpose of designing a circular shaft to with stand a given torque we must develop an equation giving the relation between Twisting moments max m shear stain produced and a quantity representing the size and shape of the cross – sectional area of the shaft.

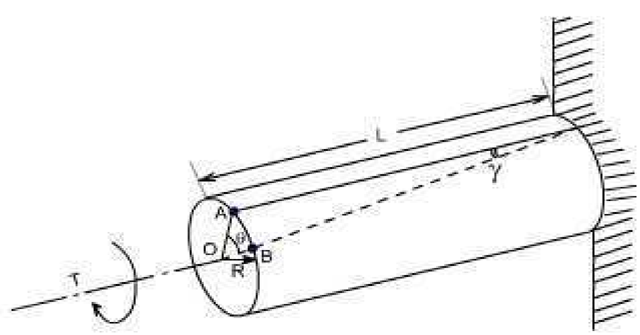


Refer to the figure shown above where a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear, the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purpose of deriving a simple theory to describe the behavior of shafts subjected to torque it is necessary make the following base assumptions.

Assumption:

The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

a. The material is elastic, follows Hook's law, with shear stress proportional to shear strain.
b. The stress does not exceed the elastic limit.
c. The circular section remains circular
d. Cross section remains plane.
e. Cross section rotates as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle θ, point A moves to B, and AB subtends an angle 'θ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

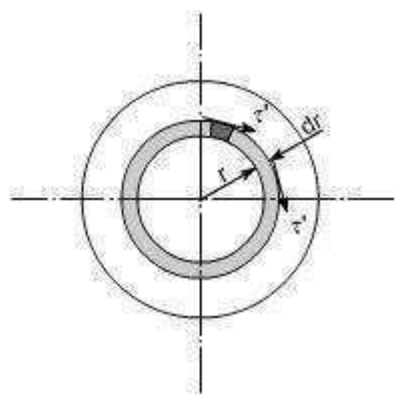
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress τ'.



The force set up on each element

= stress x area

= τ' x 2πr dr (approximately)

This force will produce a moment or torque about the center axis of the shaft.

= τ' x 2πr dr . r

= 2 τ' π. r^2. dr

The total torque T on the section, will be the sum of all the contributions.

$$T = \int_0^R 2\pi\tau' r^2 \, dr$$

Since τ' is a function of r, because it varies with radius so writing down τ' in terms of r from the equation (1).

$$\text{i.e } \tau' = \frac{G\theta.r}{L}$$

$$\text{we get } T = \int_0^R 2\pi \frac{G\theta}{L}.r^3\,dr$$

$$T = \frac{2\pi G\theta}{L}\int_0^R r^3\,dr$$

$$= \frac{2\pi G\theta}{L}.\left[\frac{R^4}{4}\right]_0^R$$

$$= \frac{G\theta}{L}.\frac{2\pi R^4}{4}$$

$$= \frac{G\theta}{L}.\frac{\pi R^4}{2}$$

$$= \frac{G\theta}{L}.\left[\frac{\pi d^4}{32}\right] \text{ now substituting } R = d/2$$

$$= \frac{G\theta}{L}.J$$

$$\text{since } \frac{\pi d^4}{32} = J \text{ the polar moment of inertia}$$

$$\text{or } \frac{T}{J} = \frac{G\theta}{L} \qquad \text{......(2)}$$

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

Where

T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia

$$= \frac{\pi d^4}{32} \text{ for solid shaft}$$

$$= \frac{\pi(D^4 - d^4)}{32} \text{ for a hollow shaft.}$$

[D = Outside diameter ; d = inside diameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

θ = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque

per radius twist i.e, k = T / θ = GJ / L

Power Transmitted by a shaft: If T is the applied Torque and ω the angular velocity of the shaft, then the power transmitted by the shaft is

$$P = T.\omega = \frac{2\pi NT}{60} = \frac{2\pi NT}{60.10^3}\,\text{kw}$$

where N= rpm

Distribution of shear stresses in circular Shafts subjected to torsion :

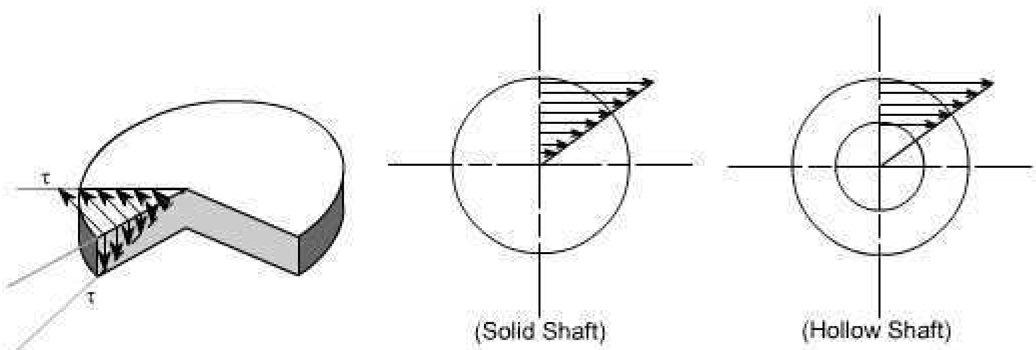
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

or

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance ‘r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress occurs on the outer surface of the

shaft where r = R The value of maximum shearing stress in the solid

circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} \cdot d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

Power Transmitted by a shaft:

In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit. Given the power required to be transmitted, speed in rpm ‘N' Torque T, the formula connecting.

These quantities can be derived as follows

$$P = T.\omega$$
$$= \frac{T.2\pi N}{60} \text{watts}$$
$$= \frac{2\pi NT}{60x10^3} (kw)$$

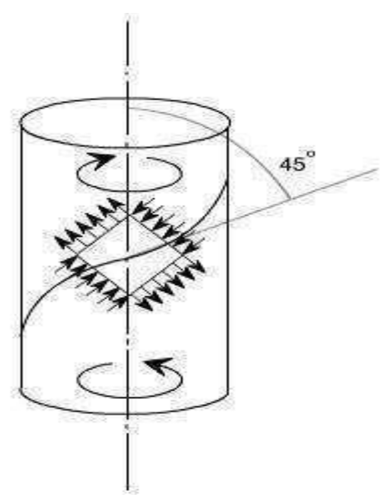
Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist.

$$k = \frac{T}{\theta}$$
$$i.e = \frac{GJ}{L}$$
$$or \ k = \frac{G.J}{L}$$

For a ductile material, the plastic flow begins first in the outer surface. For a material which is weaker in shear longitudinally than transversely – for instance a wooden shaft, with the fibres parallel to axis the first cracks will be produced by the shearing stresses acting in the axial section and they will upper on the surface of the shaft in the longitudinal direction.
In the case of a material which is weaker in tension than in shear. For instance a, circular shaft of cast iron or a cylindrical piece of chalk a crack along a helix inclined at 45^0 to the axis of shaft often occurs.

Explanation: This is because of the fact that the state of pure shear is equivalent to a state of stress tension in one direction and equal compression in perpendicular direction.

A rectangular element cut from the outer layer of a twisted shaft with sides at 45^0 to the axis will be subjected to such stresses, the tensile stresses shown will produce a helical crack mentioned.



<u>TORSION OF HOLLOW SHAFTS:</u>

From the torsion of solid shafts of circular x – section , it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

d_i = Inside diameter

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

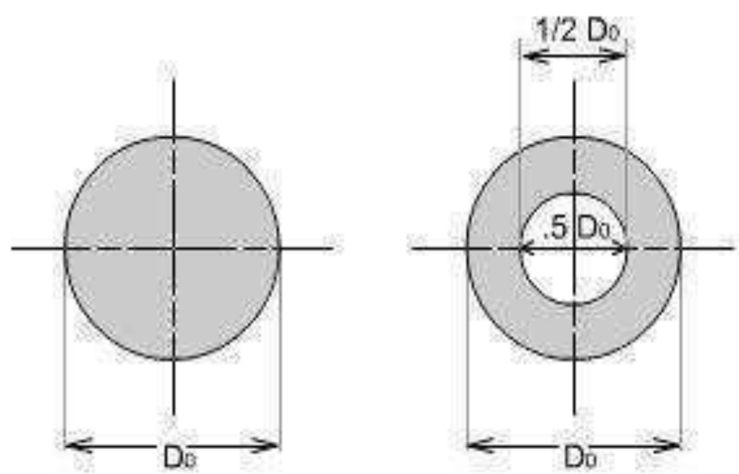
$$= \frac{16T.D_0}{\pi D_0^4\left[1 - (d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1 - (1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$

Hence by examining the equation (1) and (2) it may be seen that the τ_{max^m} in the case of hollow shaft is 6.6% larger then in the case of a solid shaft having the same outside diameter.

Reduction in weight:

Considering a solid and hollow shafts of the same length 'l' and density 'ρ' with $d_i = 1/2\ D_o$



Weight of hollow shaft

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi D_0^2}{16}\right] l \times \rho$$

$$= \frac{\pi D_0^2}{4}[1 - 1/4] l \times \rho$$

$$= 0.75 \frac{\pi D_0^2}{4} l \times \rho$$

$$\text{Weight of solid shaft} = \frac{\pi D_0^2}{4} . l . \rho$$

$$\text{Reduction in weight} = (1 - 0.75)\frac{\pi D_0^2}{4} l \times \rho$$

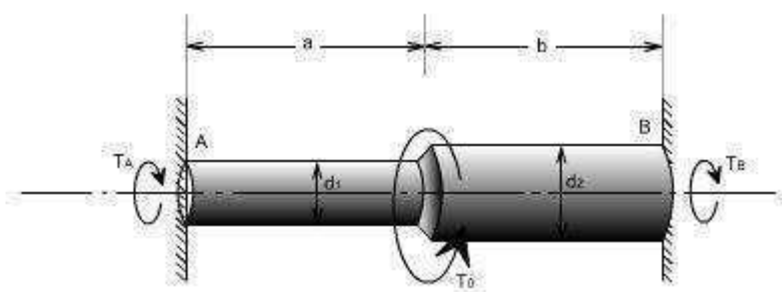
$$= 0.25 \frac{\pi D_0^2}{4} l \times \rho$$

Hence the reduction in weight would be just 25%.

Illustrative Examples :

Problem

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. Determine the angle of rotation θ_0 of the shoulder section where T_0 is applied?



Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that we can find from the statics is that the sum of two reactive torque T_A and T_B at the built – in ends of the shafts must be equal to the applied torque T_0

Thus $T_A + T_B = T_0$ ------ (1)

[From static principles]

Where T_A, T_B are the reactive torque at the built in ends A and B. whereas T_0 is the applied torque

From consideration of consistent deformation, we see that the angle of twist in each portion of the shaft must be same.

i.e $\theta_a = \theta_b = \theta_0$

Using the relation for angle of twist $\frac{T}{J} = \frac{G.\theta}{l}$

$$\text{or } \theta_A = \frac{T_A a}{J_A G}$$

$$\theta_B = \frac{T_B a}{J_B G}$$

$$\Rightarrow \frac{T_A a}{J_A G} = \frac{T_B b}{J_B G} = \theta_0 \quad \text{or} \quad \frac{T_A}{T_B} = \frac{J_A}{J_B}.\frac{b}{a} \quad (2)$$

N.B: Assuming modulus of rigidity G to be same for the

two portions So the defines the ratio of T_A and T_B

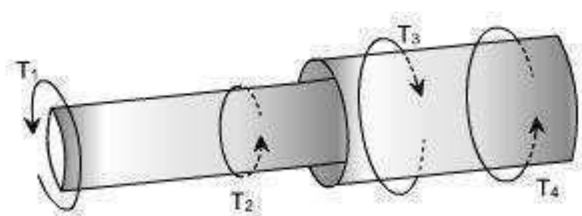
$$T_A = \frac{T_0}{1 + \frac{J_B a}{J_A b}}$$

$$T_b = \frac{T_0}{1 + \frac{J_a b}{J_b a}}$$

Using either of these values in (2) we have the angle of rotation θ_0 at the junction

$$\theta_0 = \frac{T_0.a.b}{[J_A.b + J_B.a]G}$$

Non Uniform Torsion: The pure torsion refers to torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the bar / shaft need not to be prismatic and the applied torques may vary along the length.



Here the shaft is made up of two different segments of different diameters and having torques applied at several cross sections. Each region of the bar between the applied loads between changes in cross section is in pure torsion, hence the formula's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

$$\frac{T}{J} = \frac{\tau}{r} \text{ and } \frac{T}{J} = \frac{G\theta}{L}$$

The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

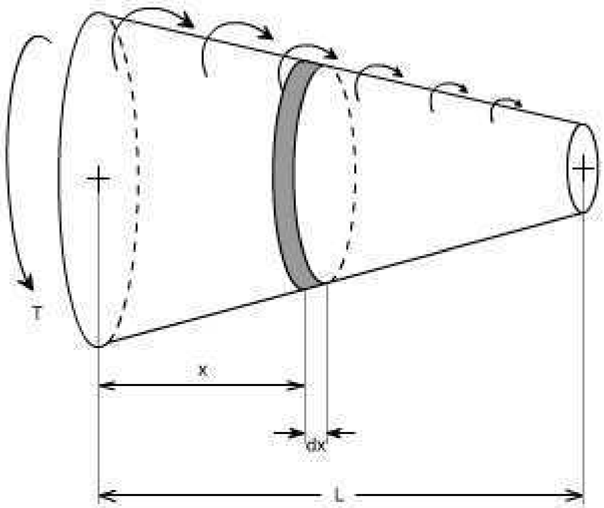
$$\theta = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

i = index for no. of parts

n = total number of parts

If either the torque or the cross section changes continuously along the axis of the bar, then the Θ

summation can be replaced by an integral sign (∫). i.e We will have to consider a differential element.



After considering the differential element, we can write $d\theta = \frac{T_x dx}{GI_x}$

Substituting the expressions for T_X and J_X at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the value of angle of twist may be determined.

$$\theta = \int_0^L d\theta = \int_0^L \frac{T_x dx}{GI_x}$$

Unsymmetrical Bending

When the plane of loads acting transversely on a beam does not contain any of the beam's axes of symmetry, the loads may tend to produce twisting as well as bending. Figure shows a horizontal channel twisting even though the vertical load H acts through the centroid of the section.

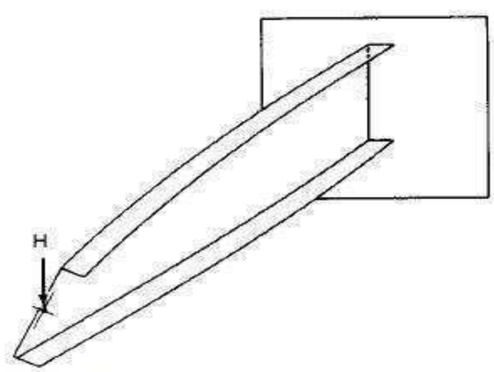
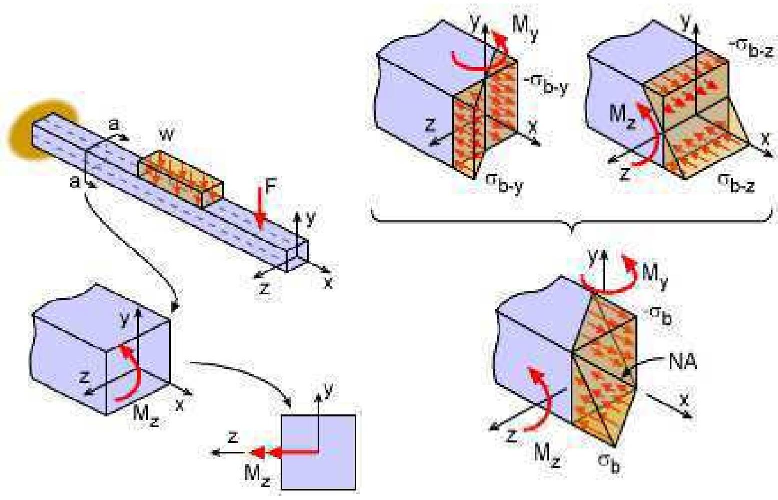


FIGURE 3.53 Twisting of a channel.

The bending axis of a beam is the longitudinal line through which transverse loads should pass to preclude twisting as the beam bends. The shear center for any section of the beam is the point in the section through which the bending axis passes.

For sections having two axes of symmetry, the shear center is also the centroid of the section. If a section has an axis of symmetry, the shear center is located on that axis but may not be at the centroid of the section.



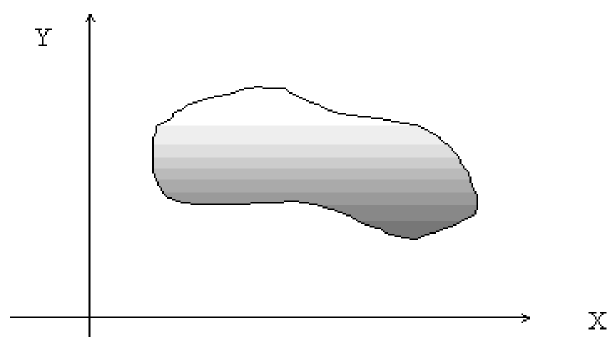
$$\sigma_b = 0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\text{Rearranging: } \frac{y}{z} = \left(\frac{M_y}{M_z}\right)\left(\frac{I_z}{I_y}\right)$$

$$\text{Since } \tan\left(\frac{y}{z}\right) = \alpha \quad \rightarrow \quad \therefore \alpha = \tan^{-1}\left[\left(\frac{M_y}{M_z}\right)\left(\frac{I_z}{I_y}\right)\right]$$

Properties of surfaces II: Second moment of area

Just as we have discussing first moment of an area and its relation with problems in mechanics, we will now describe second moment and product of area of a plane. In this lecture we look at these quantities as some mathematical entities that have been defined and solve some problems involving them. The usefulness of related quantities, called the moments of inertia and products of inertia will become clear when we deal with rotation of rigid bodies.



An area in the XY plane

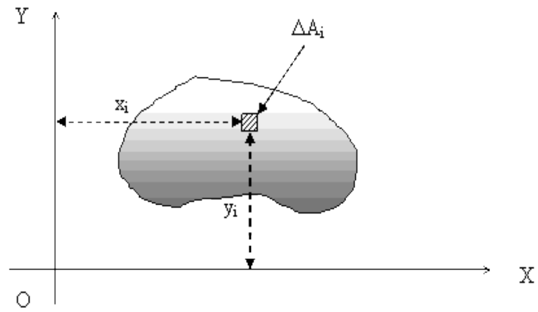
Figure 1

Let us then consider a plane area in xy plane (figure 1). The second moments of the area A is defined as

$$I_{XX} = \sum_i y_i \Delta A_i = \int_A y^2 dA$$

$$I_{YY} = \sum_i x_i \Delta A_i = \int_A x^2 dA$$

That is given a plane surface; we take a small area in it, multiply by its perpendicular distance from the x-axis and sum it over the entire area. That gives I_{XX} . Similarly I_{YY} is obtained by multiplying the small area by square of the distance perpendicular to the y-axis and adding up all contributions (see figure 2).



An element of area ΔA_i and its x- and y-coordinates

Figure 2

The product of area is defined as

$$I_{xy} = \sum_i x_i y_i \Delta A_i = \int xy\, dA$$

Where x and y are the coordinates of the small area dA . Obviously I_{XX} is the same as I_{XY}.

Product of Inertia:

Consider a three dimensional body of mass m as shown. The mass moment of inertia I about the axis O-O is defined as,

$$I = \int r^2 dm$$

Where r is the perpendicular distance of the mass element dm from the axis O-O and where the integration is over the entire body. For a given body, the mass moment of inertia is a measure of the distribution of its mass relative to the axis in in question and for that axis is a constant property of the body. In SI units, the units of measurement of inertia are kg-m^2 .

Stresses in Curved Beam

Consider a curved beam subjected to bending moment M_b as shown in the figure. The distribution of stress in curved flexural member is determined by using the following assumptions:

a. The material of the beam is perfectly homogeneous [i.e., same material throughout] and isotropic [i.e., equal elastic properties in all directions]

b. The cross section has an axis of symmetry in a plane along the length of the beam.

c. The material of the beam obeys Hooke's law.

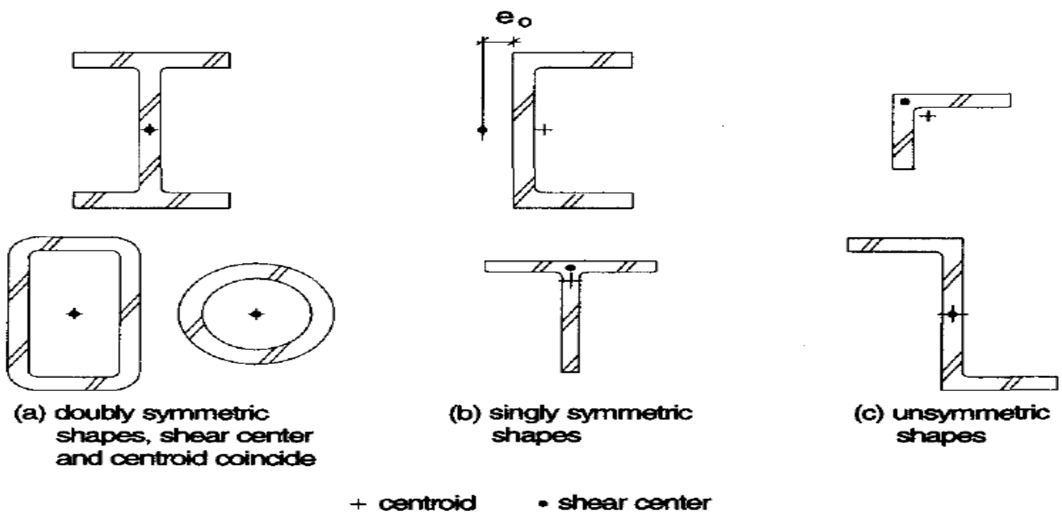
d. The transverse sections which are plane before bending remain plane after bending also.

e. Each layer of the beam is free to expand or contract, independent of the layer above or below it.

f. The Young's modulus is same both in tension and compression.

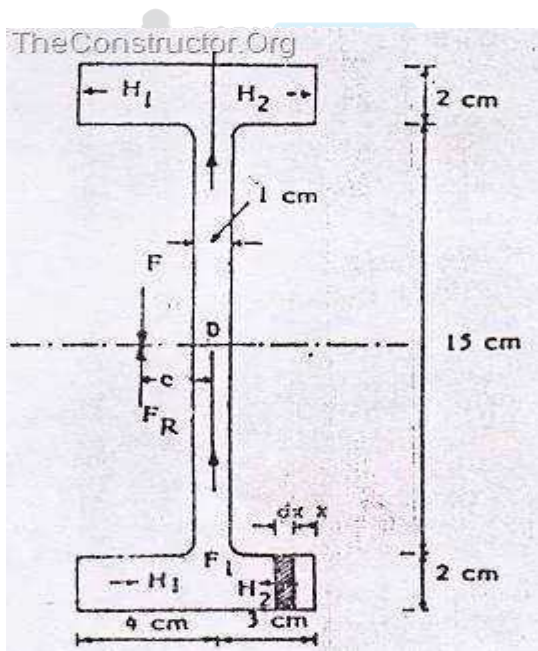
Shear Centre

If a beam is subjected to bending moments and shear force in a plane, other than the plane of geometry, which passes through the centroid of the section, then bending moment will be accompanied by twisting. In order to avoid twisting and cause bending only, the transverse forces must act through a point which may not coincide with the centroid, but will depend upon the shape of the section and such a point is termed as shear centre.



EXAMPLE

To locate the shear centre of the unsymmetrical I-beam cross section as shown in figure below:



$$H_2 = \int q dA = \int \frac{FA\bar{y}}{I \cdot t} dA$$

Here

$$= \frac{F}{I \cdot t} \int A\overline{y} dA$$

$$= \frac{F}{I \cdot t} \int_0^3 2(3 - x) \times 6.5 \times 2 \times dx$$

$$= \frac{F}{I \times 2} \int_0^3 26(3 - x) dx$$

$$= \frac{13F}{I} \left[3x - \frac{x^2}{2} \right]_0^3$$

$$= \frac{13F}{I} \left[9 - \frac{9}{2} \right] = 58.5 \frac{F}{I}$$

$$\therefore H_1 = \frac{F}{2I} \int_0^4 2(4 - x) \times 6.5 \times 2 \times dx$$

$$= \frac{13F}{I} \left[4x - \frac{x^2}{2} \right]_0^4$$

$$= \frac{13F}{I} \left[4 \times 4 - \frac{4^2}{2} \right] = \frac{104F}{I}$$

Taking moment about the point D

$$F_R \times e = 2(H_1 - H_2) \times 6.5$$

$$= 2\left(\frac{104F}{I} - \frac{58.5F}{I} \right) \times 6.5$$

$$= \frac{2F}{I}(104 - 58.5) \times 6.5$$

$$= \frac{2F}{I} \times 45.5 \times 6.5$$

$$= \frac{591.5 \times F}{I}$$

Here, $F_R = F$

$$\therefore e = \frac{591.5}{I}$$

$$I = 2\left[\frac{7 \times 2^3}{12} + 14 \times (6.5)^2 \right] + \frac{1 \times 11^3}{12}$$

$$= 1303.26 \text{cm}$$

$$\text{e} = \frac{591.5}{1303.26} = 0.45386 \text{ cm}$$

Bending of Curved Beams

The bending of beams which are initially curved. We do this by restricting ourselves to the case where the bending takes place in the plane of curvature. This happens when the cross section of the beam is symmetrical about the plane of its curvature and the bending moment acts in this plane. As we did for straight beams, we first obtain the solution assuming sections that are initially plane remain plane after bending. The resulting relation between the stress, moment and the deflection is called as Winkler-Bach formula. Then, using the two dimensional elasticity formulation, we obtain the stress and displacement field without assuming plane sections remain plane albeit for a particular cross section of a curved beam subjected to a pure bending moment or end load. We conclude by comparing both the solutions to find that they are in excellent agreement when the beam is shallow.

Before proceeding further, we would like to clarify what we mean by a curved beam. Beam whose axis is not straight and is curved in the elevation is said to be a curved beam. If the applied loads are along the y direction and the span of the beam is along the x direction, the axis of the beam should have a curvature in the xy plane.

On the hand, if the member is curved on the xz plane with the loading still along the y direction, then it is not a curved beam, as this loading will cause a bending as well as twisting of the section. Thus, a curved beam does not have a curvature in the plan. Arches are examples of curved beams.

Neutal Axis

It is helpful in understanding the bending stress if the location of the neutral axis is known, especially for unsymmetrical bending; The definition of the NA is where the bending stress is zero. Thus, if the unsymmetrical bending stress equation is forced to equal zero, the result will be the line equation for the NA, or

$$0 = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

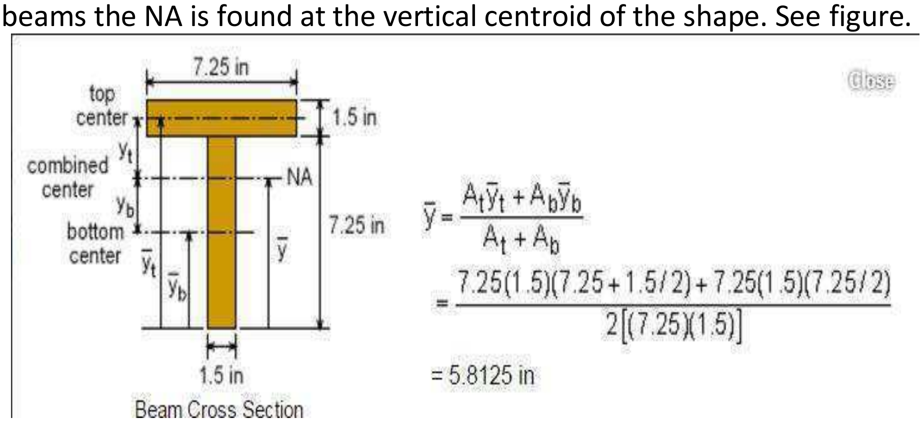
This is an equation for a line that goes through the origin. The maximum bending stress is located at the point which is furthest from the neutral axis (perpendicular direction).

The angle between the NA and the z axis is

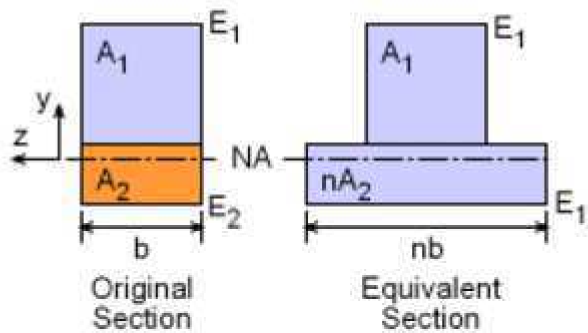
$$\theta = \tan^{-1}\left(\frac{y}{z}\right) = \tan^{-1}\left(\frac{M_y I_z}{M_z I_y}\right)$$

Locating the Neutral Axis (NA) can be very complicated.

In the simplest of beams the NA is found at the vertical centroid of the shape. See figure.



For composite beams, those made of different materials adjustments need to be made. See figure 2 below. An "equivalent Section" needs to be drawn, where the width of one of the materials is multiplied by the factor "n". n is the ratio of the two materials Young's moduli. In the example below n>1 so the area increased. Then the centroid of the Equivalent section is found as described above.



Equivalent Area Method Cross Section

$$n = E_2 / E_1$$

